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L-fuzzy Prime and Maximal Congruences in Almost Distributive Lattices

Natnael Teshale Amare[1] and S. Nageswara Rao[2]

[1]Department of Mathematics, University of Gondar, Ethiopia

[2]Department of Mathematics, Mallareddy College of Engineering, Hyderabad, India-500014 *

Correspondence: yenatnaelteshale@gmail.com

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ABSTRACT

In this paper, we define and describe the concept of L-fuzzy prime and maximal congruences in an Almost Distributive lattice (ADL) and discuss its characteristics. Mainly, we establish a one-to-one correspondence between prime (maximal) L-fuzzy congruences of an ADL and the pairs (P, α), where P is a prime (maximal) congruence of an ADL and α is a prime element (dual atom) in a frame, yields the prime (maximal) L-fuzzy congruences of all given ADL. Furthermore, we examine the relationship between prime (maximal) L-fuzzy congruence and L-fuzzy prime (maximal) congruence on an ADL, proving through counter examples that the converse is not true.

Keywords: Almost Distributive Lattice (ADL), L-fuzzy congruence, maximal L-fuzzy congruence, prime L-fuzzy congruence, L-fuzzy maximal congruence and L-fuzzy prime congruence.

1 Introduction

The significance of prime ideals is crucial in exploring the structural theory of distributive lattices in a broad sense, particularly in the context of Boolean algebras. Jayaram (1986) and Pawar (2012) have introduced the notion of prime $\alpha-$ideals and space of prime $\alpha-$ideals in distributive lattice, respectively. In 1981, Swamy and Rao . (1981) was introduced the concept of an Almost Distributive Lattice (ADL) as a common abstraction to most of the existing ring theoretic generalizations of a Boolean algebra. Consequently, several researchers have conducted studies on almost distributive lattices (ADLs), focusing on topics such as pseudo-complementation in ADLs,Swamy (200). Currently, Norham et al. (2023) discussed $\sigma-$ideals in MS-ADL. Zadeh (1965). initially defined a fuzzy subset of a set X as a function mapping elements of X to real numbers in the interval $[0, 1]$. Goguen (1967) extended this by replacing the valuation set $[0, 1]$ with a complete lattice L, aiming to provide a more comprehensive exploration of fuzzy set theory through fuzzy sets. Concurrently, Liu (1982). explored the definition and application of fuzzy subrings and fuzzy ideals within rings. Similarly, Rosenfeld (1971) utilized this idea within group theory to establish the theoretical foundation for fuzzy subgroups. Subsequently, numerous scholars have dedicated their research to the exploration of fuzzy subrings and ideals within rings and lattices [refer to Lehmke, 1997; Malik & Mordeson, 1992; Mukharjee& Sen, 1984; Swamy & Raju, 1991, 1998; Swamy & Swamy, 1988]. The concept of Almost Distributive Lattices (ADLs) was later introduced by Swamy and Rao (1981). Building on this, Swamy, Raj and Natnael (2017) have proposed the concepts of fuzzy ideals of an ADL and the notion of fuzzy congruence relation on an ADL was introduced by Alaba et al. (2017). Currently, Natnael have introduced the concept of L-weakly 1-absorbing prime ideals and filters.

In this paper, we extend the results of prime and maximal concepts of fuzzy ideals in an ADL (Raj, Amare & Swamy, 2018a,2018a) to the case of congruence relations of ADLs. Section 3 focuses on the discussion of an L-fuzzy prime congruence on an ADL. Our primary achievement has been proving a characterization theorem for prime L-fuzzy congruence, viewed simply as a prime element within the lattice of L-fuzzy congruence. This theorem asserts that an L-fuzzy congruence ϕ on an ADL is prime if and only if a prime congruence θ on an ADL exists and a prime element β in a frame such that $\phi = \beta_{\phi}$. This demonstrates a direct correlation between prime L-fuzzy congruence on an ADL and pairs (θ, β), where θ is a prime congruence on an ADL and β is a prime element in L. Additionally, we introduce a less stringent version of L-fuzzy prime congruence compared to the prime L-fuzzy congruence on A. In section 4, we delve into the concept of an L-fuzzy maximal congruence on an ADL as a proper L-fuzzy congruence, with each of its β-cuts either being a maximal congruence on A or the entirety of an ADL. Finally, we establish a direct correlation between maximal L-fuzzy congruence on an ADL and pairs (θ, β), where β is a dual atom in L and θ is a maximal congruence on an ADL.

Throughout this paper, A stands for an ADL $(A, \wedge, \vee, 0)$ with a maximal element and L stands for a complete lattice $(L, \wedge, \vee, 0, 1)$ satisfying the infinite meet distributive law and this type of a lattice is called a frame.

2 Preliminaries

In this section, we recall some definitions and basic results mostly taken from (0,) and (0,).

Definition 2.1. An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following conditions for all a, b and $c \in A$.

1. $0 \wedge a = 0$
2. $a \vee 0 = a$
3. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
4. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
5. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
6. $(a \vee b) \wedge b = b$

Any bounded below distributive lattice is an ADL. Any nonempty set X can be made into an ADL which is not a lattice by fixing an arbitrarily chosen element 0 in X and by defining the binary operations $\wedge$ and $\vee$ on X by

$$a \wedge b = \begin{cases} 0, & \text{if } a = 0 \\ b, & \text{if } a \neq 0 \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } a \neq 0. \end{cases}$$

This ADL $(X, \wedge, \vee, 0)$ is called a discrete ADL.

Definition 2.2. Let $A = (A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define $a \leq b$ if $a = a \wedge b$ ($\Leftrightarrow a \vee b = b$). Then $\leq$ is a partial order on A with respect to which 0 is the smallest element in A.

Theorem 2.3. *The following hold for any a, b and c in an ADL A.*

(1) $a \wedge 0 = 0 = 0 \wedge a$ *and* $a \vee 0 = a = 0 \vee a$

(2) $a \wedge a = a = a \vee a$

(3) $a \wedge b \leq b \leq b \vee a$

(4) $a \wedge b = a \Leftrightarrow a \vee b = b$

(5) $a \wedge b = b \Leftrightarrow a \vee b = a$

(6) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *(i.e.,* $\wedge$ *is associative)*

(7) $a \vee (b \vee a) = a \vee b$

(8) $a \leq b \Rightarrow a \wedge b = a = b \wedge a \; \big(\Leftrightarrow a \vee b = b = b \vee a \big)$

(9) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$

(10) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(11) $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$

(12) $a \wedge b = \inf\{a, b\} \Leftrightarrow a \wedge b = b \wedge a \Leftrightarrow a \vee b = \sup\{a, b\}$.

Definition 2.4. Let I be a non empty subset of an ADL A. Then I is called an ideal of A if $a, b \in I \Rightarrow a \vee b \in I$ and $a \wedge x \in I$ for all $x \in A$.

A proper ideal P of an ADL A is said to be prime if for any $x, y \in A$, $x \wedge y \in P$ implies that $x \in P$ or $y \in P$.

An element $m \in A$ is said to be maximal if, for any $x \in A$, $m \leq x$ implies $m = x$. It can be easily observed that m is maximal if and only if $m \wedge x = x$ for all $x \in A$.

Definition 2.5. Let L be a frame. An element β in L is a prime element in L if for any γ and $\alpha \in L$ such that $\gamma \wedge \alpha \leq \beta$, then either $\gamma \leq \beta$ or $\alpha \leq \beta$.

Definition 2.6. A fuzzy relation ϕ on an ADL A is called a fuzzy congruence relation on A, if the following are satisfied:

(1) $\phi(x, x) = 1$, for all $x \in A$,

(2) $\phi(x, y) = \phi(y, x)$,for all $x, y \in A$,

(3) $\phi(x, y) \wedge \phi(y, z) \leq \phi(x, z)$, for all $x, y, z \in A$,

(4) $\phi(x, y) \wedge \phi(z, t) \leq \phi(x \vee z, y \vee t) \wedge \phi(x \wedge z, y \wedge t)$, for all $x, y, z, t \in A$.

3 L-fuzzy Prime Congruences

In this section, we present the concept of L-fuzzy prime congruence on an ADL A which is weaker than that a prime L-fuzzy congruence on A. Initially, we discuss a prime L-fuzzy congruence on A as simply a prime element in the lattice of L-fuzzy congruence.

Here, we have mainly proved a characterization theorem for prime L-fuzzy congruence which states that an L-fuzzy congruence ϕ on A is prime if and only if there is a prime congruence θ on A and a prime element $\beta \in L$ such that $\phi = \beta_\theta$.

Recall from (0,) that for any set A, an L-fuzzy relation ϕ of $A \times A$ is called an L-fuzzy congruence on A. First, we have the following.

Definition 3.1. A congruence θ on A is prime if $\theta \neq A \times A$ and, for any congruences ψ and ζ on A,

$$\psi \cap \zeta \subseteq \theta \Rightarrow \psi \subseteq \theta \text{ or } \zeta \subseteq \theta.$$

We denote the zero element of the lattice of all congruence relations on A by Δ_A. That is, $\Delta_A = \{(x, y) \in A \times A : x = y\}$.

The following definition is analogous to that of a prime congruence on A.

Definition 3.2. A proper L-fuzzy congruence ϕ on A is called prime if for any L-fuzzy congruences θ and ψ on A,

$$\eta \wedge \psi \leq \phi \Rightarrow \eta \leq \phi \text{ or } \psi \leq \phi.$$

First let us recall that, for any crisp congruence θ on A and an element β in L, define $\beta_\theta : A \to L$ by

$$\beta_\theta(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \theta \\ \beta & \text{if } (x, y) \notin \theta \end{cases}$$

and that β_θ is called the β-level L-fuzzy congruence.

Theorem 3.3. *Let θ be a congruence on A and $\beta \in L$. Then the $\beta-$level fuzzy congruence β_θ is a prime L-fuzzy congruence on A if and only if θ is a prime congruence on A and β is a prime element in L.*

Proof. First we observe that β_θ is a proper L-fuzzy congruence on A iff $\theta \neq A \times A$ and $\beta \neq 1$. Therefore, we can assume that $\theta \neq A \times A$ and $\beta \neq 1$. Then β_θ is proper. Now, assume that θ is a prime congruence on A and β is a prime element in L. Let ϕ and η be L-fuzzy congruences on A such that $\phi\beta_\theta$ and $\eta\beta_\theta$. Then there exists elements $a, b, c, d \in A$ such that $\phi(a, b)\beta_\theta(a, b)$ and $\eta(a, b)\beta_\theta(c, d)$. Then $(a, b) \notin \theta$, $(c, d) \notin \theta$, $\phi(a, b)\beta$ and $\eta(c, d)\beta$. Since β is a prime element in L, we get that $\phi(a, b) \wedge \eta(c, d)\beta$... $(*)$ Let $\phi_{\phi(a,b)}$ and $\eta_{\eta(c,d)}$ be the congruence on A. Also, since θ is a prime congruence on A and $(a, b) \notin \theta$ and

$(c,d) \notin \theta$, we get that $\phi_{\phi(a,b)} \cap \eta_{\eta(c,d)}\theta$ and hence we choose $(x,y) \in \phi_{\phi(a,b)} \cap \eta_{\eta(c,d)}$ such that $(x,y) \notin \theta$. Thus $(x,y) \in \phi_{\phi(a,b)}$ and hence $\phi(a,b) \leq \phi(x,y)$. Similarly, $(x,y) \in \eta_{\eta(c,d)}$ and hence $\eta(c,d) \leq \eta(x,y)$. Therefore, $\eta(a,b) \wedge \eta(c,d) \leq \phi(x,y) \wedge \eta(x,y) = (\phi \wedge \eta)(x,y)$. From (*), it follows that, $(\phi \wedge \eta)(x,y)\beta = \beta_\theta(x,y)$ (since, $(x,y) \notin \theta$). Therefore, $\phi \wedge \eta \beta_\theta$. Thus, β_θ is a prime L-fuzzy congruence on A. Conversely suppose that β_θ is prime. We already have that $\theta \neq A \times A$ and $\beta \neq 1$. Let θ_1 and θ_2 be congruences on A such that $\theta_1 \cap \theta_2 \subseteq \theta$. Then $\beta_{\theta_1} \cap \beta_{\theta_2} = \beta_{\theta_1 \cap \theta_2} \leq \beta_\theta$ and hence $\beta_{\theta_1} \leq \beta_\theta$ or $\beta_{\theta_2} \leq \beta_\theta$, which implies that $\theta_1 \subseteq \theta$ or $\theta_2 \subseteq \theta$. Therefore, θ is a prime congruence on A. Next, let α and $\gamma \in L$ such that $\alpha \wedge \gamma \leq \beta$. Then $\alpha_\theta \wedge \gamma_\beta = (\alpha \wedge \gamma)_\beta \leq \beta_\theta$ and hence $\alpha_\theta \leq \beta_\theta$ or $\gamma_\theta \leq \beta_\theta$. From this, it follows that, $\alpha \leq \beta$ or $\gamma \leq \beta$. Therefore, β is a prime element in L. □

Let us recall that for any relation θ on A, the characteristic map $\chi_\theta : A \times A \to L$ by

$$\chi_\theta(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \theta \\ 0 & \text{if } (x,y) \notin \theta. \end{cases}$$

Also, θ is a congruence on A if and only if χ_θ is an L-fuzzy congruence on A.

Theorem 3.4. *A proper L-fuzzy congruence ϕ on A is prime if and only if the following are satisfied.*

(1) ϕ assumes exactly two values

(2) for any a and $b \in A$ such that $\phi(a,b) < 1$, $\phi(a,b)$ is a prime element in L

(3) $\phi_1 = \{(a,b) \in A \times A : \phi(a,b) = 1\}$ is a prime congruence on A.

Proof. Suppose ϕ is a prime L-fuzzy congruence on A. Then we can prove (1), (2) and (3). (1). Suppose ϕ assumes more than two values, say, β and γ, other than 1. Then there exists $a,b,c,d \in A$ such that $\phi(a,b) = \gamma$ and $\phi(c,d) = \beta$. Now, consider the L-fuzzy congruences $\chi_{\phi_{\phi(c,d)}}$ and β_θ defined by

$$\chi_{\phi_{\phi(c,d)}}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \phi_{\phi(c,d)} \\ 0 & \text{if } (x,y) \notin \phi_{\phi(c,d)} \end{cases}$$

and

$$\beta_\theta(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \theta \\ \beta & \text{if } (x,y) \notin \theta. \end{cases}$$

Clearly, $\chi_{\phi_{\phi(c,d)}}$ and β_θ are L-fuzzy congruences on A. Since, $\phi(c,d) = \beta$, then $(c,d) \in \phi_\beta$ and hence ϕ_β is a congruence on A. Therefore, $\phi_{\phi(c,d)} \subseteq \phi_\beta$ and hence $\beta = \phi(c,d) \leq \phi(x,y)$, for all $(x,y) \in \phi_{\phi(c,d)}$. If $(x,y) \in \phi_{\phi(c,d)}$, then $(\chi_{\phi_{\phi(c,d)}} \wedge \beta_\theta)(x,y) = \chi_{\phi_{\phi(c,d)}}(x,y) \wedge \beta_\theta(x,y) = 1 \wedge \beta_\theta(x,y) \leq \phi(x,y)$ (since $\beta \leq \phi(x,y)$) and if, $(x,y) \notin \phi_{\phi(c,d)}$, then $(\chi_{\phi_{\phi(c,d)}} \wedge \beta_\theta)(x,y) = \chi_{\phi_{\phi(c,d)}}(x,y) \wedge \beta_\theta(x,y) = 0 \leq \phi(x,y)$. Therefore, $\chi_{\phi_{\phi(c,d)}} \wedge \beta_\theta \leq \phi$. Since ϕ is a prime L-fuzzy congruence, then either $\chi_{\phi_{\phi(c,d)}} \leq \phi$ or $\beta_\theta \leq \phi$. But $\chi_{\phi_{\phi(c,d)}} \phi$, since $\phi(c,d) < 1$. Thus, $\beta_\theta \leq \phi$. In particular, since $(a,b) \notin \theta$, we get that $\beta = \beta_\theta(a,b) \leq \phi(a,b) = \gamma$. Therefore, $\beta \leq \gamma$. Similarly, we can prove that $\gamma \leq \beta$. Thus, $\beta = \gamma$. Since ϕ is proper, there at least $x, y \in A$ such that $\phi(x,y) < 1$. Thus ϕ has exactly two values.

(2). Let $a, b \in A$ such that $\phi(a,b) < 1$. Let β and γ be arbitrary elements in L such that $\beta \wedge \gamma \leq \phi(a,b)$. Consider the L-fuzzy congruences β_θ and γ_β defined by,

$$\beta_\theta(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \theta \\ \beta & \text{if } (x,y) \notin \theta \end{cases}$$

and

$$\gamma_\theta(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \theta \\ \gamma & \text{if } (x,y) \notin \theta. \end{cases}$$

Note that, by (1), $\phi(x,y) = \phi(a,b)$ whenever $\phi(x,y) < 1$ and hence $\phi(x,y) = \phi(a,b)$, for all $(x,y) \in A \times A - \theta$. Now, $\beta_\theta \wedge \gamma_\theta = (\beta \wedge \gamma)_\beta \subseteq \phi$, since $\beta \wedge \gamma \leq \phi(a,b)$. Again, since ϕ is prime, $\beta_\theta \leq \phi$ or $\gamma_\theta \leq \phi$. Therefore, $\beta_\theta(a,b) \leq \phi(a,b)$ or $\gamma_\theta(a,b) \leq \phi(a,b)$. But, since $\phi(a,b) < 1$ and $(a,b) \notin \theta$, we get that $\beta \leq \phi(a,b)$ or $\gamma \leq \phi(a,b)$. Thus $\phi(a,b)$ is a prime element in L.

(3). Let $\theta = \{(a,b) \in A \times A : \phi(a,b) = 1\}$. First note that θ is a proper congruence on A. Let θ_1 and θ_2 be congruences on A such that $\theta_1 \cap \theta_2 \subseteq \theta$. Then $\chi_{\theta_1} \cap \chi_{\theta_2} = \chi_{\theta_1 \cap \theta_2} \leq \chi_\theta \leq \phi$. Since ϕ is prime, either $\chi_{\theta_1} \leq \phi$ or $\chi_{\theta_2} \leq \phi$, which implies that $\theta_1 \subseteq \theta$ or $\theta_2 \subseteq \theta$. Thus θ is a prime congruence on A. Conversely suppose that conditions (1), (2) and (3) are satisfied. Since ϕ is an L-fuzzy congruence on A. By (1), 1 is a value of ϕ and ϕ assumes exactly two values. Let β be the value of ϕ other than 1. By (2), β is a prime element in L. Also, let $\theta = \{(a,b) \in A \times A : \phi(a,b) = 1\}$. Then by (3), θ is a prime congruence on A. Now, for any $a, b \in A$, we have

$$\phi(a,b) = \begin{cases} 1 & \text{if } (a,b) \in \theta \\ \beta & \text{if } (a,b) \notin \theta \end{cases}$$

and hence $\phi = \beta_\theta$, which is a prime L-fuzzy congruence on A. □

As a consequence of Theorems 3.3 and 3.4, we have the following.

Theorem 3.5. *Let ϕ be an L-fuzzy congruence on A. Then ϕ is prime if and only if $\phi = \beta_\theta$ for some prime element β in L and a prime congruence θ on A.*

Corollary 3.6. *Let θ be a congruence on A. Then χ_θ is a prime L-fuzzy congruence on A if and only if θ is a prime congruence on A and 0 a $\wedge$-prime element in L.*

Theorem 3.7. *For any β and $\gamma \in L$ and proper congruences θ and ψ on A,*
$\beta_\theta \le \gamma_\psi \Leftrightarrow \theta \subseteq \psi$ and $\beta \le \gamma$, where β_θ and γ_ψ are β-level and γ-level fuzzy congruences on A corresponding to θ and ψ.

Proof. Suppose that $\beta_\theta \le \gamma_\psi$. Then for any $(a,b) \in \theta$, $1 = \beta_\theta(a,b) \le \gamma(a,b)$ and hence $\gamma_\psi(a,b) = 1$. So that $(a,b) \in \psi$. Therefore, $\theta \subseteq \psi$. For any $(c,d) \in A \times A - \psi$, we have $(c,d) \notin \theta$ and $\beta = \beta_\theta(c,d) \le \gamma_\psi(c,d) = \gamma$. Thus, $\beta \le \gamma$. Conversely suppose that $\theta \subseteq \psi$ and $\beta \le \gamma$. For any $(a,b) \in A \times A$, $(a,b) \in \theta \Rightarrow (a,b) \in \psi$ and $\beta_\theta(a,b) = 1 = \gamma_\psi(a,b)$ and $(a,b) \notin \theta \Rightarrow \beta_\theta(c,d) = \beta \le \gamma$ or $1 = \gamma_\psi(c,d)$ and hence $\beta_\theta \le \gamma_\psi$. □

Theorem 3.8. *$(\phi,\beta)\beta_\phi$ establishes a one-to-one correspondence between the pair (ϕ,β), where ϕ is a prime congruence on A and β a prime element in L, and the prime L-fuzzy congruence on A.*

In the following, we introduce the notion of L-fuzzy prime congruence on an ADL. First, let us recall that the $\alpha-$cut ϕ_α defined by

$$\phi_\alpha = \{(x,y) \in A \times A : \alpha \le \phi(x,y)\}.$$

Definition 3.9. A proper L-fuzzy congruence ϕ on A is called an L-fuzzy prime congruence on A, if for each $\alpha \in L$, $\phi_\alpha = A \times A$ or ϕ_α is a prime congruence on A.

Theorem 3.10. *Let ϕ be an L-fuzzy prime congruence on A. Then $\phi(A \times A)$ is a chain in L; that is, for any $a,b,c,d \in A$*

$$\text{either } \phi(a,b) \le \phi(c,d) \text{ or } \phi(c,d) \le \phi(a,b).$$

Proof. Let ϕ be an L-fuzzy prime congruence on A. Then the α-cut ϕ_α is either $A \times A$ or a prime congruence on A, for all $\alpha \in L$. Let $a,b,c,d \in A$ and put $\alpha = \phi(a,b) \vee \phi(c,d)$. Consider the congruences $\phi_{\phi(a,b)}$ and $\phi_{\phi(c,d)}$ corresponding to the pairs (a,b) and (c,d) in A respectively. Then $(x,y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)} \Rightarrow (x,y) \in \phi_{\phi(a,b)}$ and $(x,y) \in \phi_{\phi(c,d)}$
$\Rightarrow \phi(a,b) \le \phi(x,y)$ and $\phi(c,d) \le \phi(x,y)$

$\Rightarrow \phi(a,b) \vee \phi(c,d) \leq \phi(x,y)$

$\Rightarrow \alpha \leq \phi(x,y)$

$\Rightarrow (x,y) \in \phi_\alpha$.

Therefore, $\phi_{\phi(a,b)} \cap \phi_{\phi(c,d)} \subseteq \phi_\alpha$ and hence $\phi_{\phi(a,b)} \subseteq \phi_\alpha$ or $\phi_{\phi(c,d)} \subseteq \phi_\alpha$.

Now, $\phi_{\phi(a,b)} \subseteq \phi_\alpha \Rightarrow (a,b) \in \phi_\alpha$

$\Rightarrow \alpha \leq \phi(a,b)$

$\Rightarrow \phi(a,b) \vee \phi(c,d) \leq \phi(a,b)$

$\Rightarrow \phi(c,d) \leq \phi(a,b)$.

Similarly, $\phi(a,b) \leq \phi(c,d)$ (if $\phi_{\phi(c,d)} \subseteq \phi_\alpha$). Thus either $\phi(a,b) \leq \phi(c,d)$ or $\phi(c,d) \leq \phi(a,b)$. Therefore, $\phi(A \times A)$ is a chain in L. □

The converse of the above theorem is not true; For, consider the following example. Let I be a proper ideal of A and $L = \{0, \alpha, 1\}$ with $0 < \alpha < 1$. Define ϕ on $A \times A$ by

$$\phi(x,y) = \begin{cases} 1 & \text{if } x = y \\ \alpha & \text{if } x \neq y \text{ and } x \vee y \in I \\ 0 & \text{if } x \vee y \notin I. \end{cases}$$

Then for each $\beta \in L$, ϕ_β is given by

$$\phi_\beta = \begin{cases} A \times A & \text{if } \beta = 0 \\ \theta & \text{if } 0 < \beta \leq \alpha \\ \Delta_A & \text{if } \beta\alpha, \end{cases}$$

where Δ_A is the zero congruence on A and $\theta = \{(x,y) \in A \times A : x \vee y \in I\}$. Then ϕ is an L-fuzzy congruence on A, since ϕ_β is a congruence on A, for each $\beta \in L$. Also, $\phi(A \times A) = \{0, \alpha, 1\}$ is a chain in L, while ϕ is not an L-fuzzy prime congruence on A if we take I to be not a prime ideal of A.

For any $a, b \in A$, $\phi_{\phi(a,b)}$ is the smallest congruence on A containing the pair (a,b).

Theorem 3.11. *Let ϕ be a proper L-fuzzy congruence on A such that $\phi(A \times A)$ is a chain in L. Then ϕ is an L-fuzzy prime congruence on A if and only if, for any $a, b, c, d \in A$,*

$$\phi(a,b) \vee \phi(c,d) = \wedge\{\phi(x,y) : (x,y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)}\}.$$

Proof. For any $a, b, c, d \in A$, $\phi(a,b) \vee \phi(c,d) = max\{\phi(a,b), \phi(c,d)\}$ (since $\phi(A \times A)$ is a chain in L). Suppose that ϕ is an L-fuzzy prime congruence on A. Let $\theta = \{\phi(x,y) :$

$(x, y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)}\}$. Then $\phi_{\phi(a,b)} \vee \phi_{\phi(c,d)} \leq \phi(x, y)$, for all $(x, y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)}$ and hence $\phi_{\phi(a,b)} \vee \phi_{\phi(c,d)}$ is a lower bound in L. Let β be another lower bound of θ in L. Then $\beta \leq \phi(x, y)$, for all $(x, y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)}$ and hence $\phi_{\phi(a,b)} \cap \phi_{\phi(c,d)} \subseteq \phi_\beta$. Since ϕ is an L-fuzzy prime congruence on A, it follows that $\phi_{\phi(a,b)} \subseteq \phi_\beta$ or $\phi_{\phi(c,d)} \subseteq \phi_\beta$; that is $(a, b) \in \phi_\beta$ or $(c, d) \in \phi_\beta$. Therefore, $\beta \leq \phi(a, b)$ or $\beta \leq \phi(c, d)$ and hence $\beta \leq \phi(a, b) \vee \phi(c, d)$. Thus $\phi(a, b) \vee \phi(c, d)$ is the infimum of θ in L. Conversely, suppose that the given condition is satisfied. Let $\beta \in L$ and θ and ψ be congruences on A such that $\theta\phi_\beta$ and $\psi\phi_\beta$. Then there exists $(a, b) \in \theta$ and $(c, d) \in \psi$ such that $(a, b) \notin \phi_\beta$ and $(c, d) \notin \phi_\beta$. That is, $\beta\phi(a, b)$ and $\beta\phi(c, d)$ and hence $\beta max\{\phi(a, b), \phi(c, d)\} = \phi(a, b) \vee \phi(c, d)$. Since $\phi(a, b) \vee \phi(c, d) = \wedge\{\phi(x, y) : (x, y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)}\}$, it follows that $\beta\phi(x, y)$ for some $(x, y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)} \subseteq \theta \cap \psi$ and hence $\theta \cap \psi\phi_\beta$. Therefore, $\phi_\beta = A \times A$ or ϕ_β is a prime congruence on A. Thus ϕ is an L-fuzzy prime congruence. □

In the following theorem, we discuss the inter relationship between prime L-fuzzy congruence and L-fuzzy prime congruence on A.

Theorem 3.12. *Every prime L-fuzzy congruence on A is an L-fuzzy prime congruence on A and the converse of this is not true.*

Proof. Let ϕ be a prime L-fuzzy congruence on A. Then by Theorem 3.5, there exists a prime congruence θ on A and a prime element β in L such that $\phi = \beta_\theta$. Since $\beta < 1$, ϕ is proper. Also, since $\phi(A \times A) = \{\beta, 1\}$, $\phi(A \times A)$ is a chain in L. We use Theorem 3.12, to prove that ϕ an L-fuzzy prime congruence on A. Let $a, b, c, d \in A$. Now,
$(a, b) \in \theta$ or $(c, d) \in \theta \Rightarrow \phi(a, b) = 1$ or $\phi(c, d) = 1$ and $\phi_{\phi(a,b)} \cap \phi_{\phi(c,d)} \subseteq \theta$
$\Rightarrow \phi(a, b) \vee \phi(c, d) = 1$ and $\phi(x, y) = 1$. for all $(x, y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)}$
$\Rightarrow \phi(a, b) \vee \phi(c, d) = 1 = \wedge\{\phi(x, y) : (x, y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)}\}$ and
$(a, b) \notin \theta$ or $(c, d) \notin \theta \Rightarrow \phi(a, b) = \beta = \phi(c, d)$ and $\phi_{\phi(a,b)} \cap \phi_{\phi(c,d)}\theta$
$\Rightarrow \phi(a, b) \vee \phi(c, d) = \beta$ and $(x, y) \notin \theta$, for some $(x, y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)}$
$\Rightarrow \phi(a, b) \vee \phi(c, d) = \beta = \phi(x, y)$, for some $(x, y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)}$
$\Rightarrow \phi(a, b) \vee \phi(c, d) = \beta = \wedge\{\phi(x, y) : (x, y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)}\}$.
Thus, $\phi(a, b) \vee \phi(c, d) = \wedge\{\phi(x, y) : (x, y) \in \phi_{\phi(a,b)} \cap \phi_{\phi(c,d)}\}$. Therefore, by Theorem 3.12, ϕ is an L-fuzzy prime congruence on A. □

Let P be a prime ideal of A and $L = \{0, \alpha, 1\}$ with $0 < \alpha < 1$. Define ϕ on $A \times A$ by

$$\phi(x,y) = \begin{cases} 1 & \text{if } x = y \\ \alpha & \text{if } x \neq y \text{ and } x \vee y \in P \\ 0 & \text{if } x \vee y \notin P. \end{cases}$$

Then for each $\gamma \in L$, ϕ_γ is given by

$$\phi_\gamma = \begin{cases} A \times A & \text{if } \gamma = 0 \\ \theta & \text{if } 0 < \gamma \leq \alpha \\ \Delta_A & \text{if } \gamma\alpha, \end{cases}$$

where Δ_A is the zero congruence on A and $\theta = \{(x,y) \in A \times A : x \vee y \in P\}$. Clearly, ϕ_γ is a congruence on A, for each $\gamma \in L$. Therefore, ϕ is an L-fuzzy congruence on A. From the given above, ϕ has more than two values. Thus ϕ is not a prime L-fuzzy congruence on A. However, ϕ is an L-fuzzy prime congruence on A, since, for each γ, ϕ_γ is either $A \times A$ or a prime congruence on A. Note here that the above θ and Δ_A are prime congruences on A; for, let θ_1 and θ_2 be any congruence on A. Then

$\theta_1\Delta_A$ and $\theta_2\Delta_A \Rightarrow$ There exists $x, y, z, t \in A$ such that $(x,y) \in \theta_1$ and $(z,t) \in \theta_2$, $x \neq$
y and $z \neq t$

$\Rightarrow \Big(x \wedge (z \vee t), y \wedge (z \vee t)\Big) \in \theta_1$ and $(z \vee t, 0) \in \theta_2$

$\Rightarrow \Big(x \wedge (z \vee t), y \wedge (z \vee t)\Big) \in \theta_1$, $\Big(x \wedge (z \vee t), 0\Big) \in \theta_2$ and $\Big(y \wedge (z \vee t), 0)\Big) \in \theta_2$

$\Rightarrow \Big(x \wedge (z \vee t), y \wedge (z \vee t)\Big) \in \theta_1 \cap \theta_2, \;\; x \wedge (z \vee t) \neq y \wedge (z \vee t)$

$\Rightarrow \theta_1 \cap \theta_2\Delta_A$.

Thus Δ_A is a prime congruence on A. Similarly, θ is a prime congruence on A.

4 L-fuzzy Maximal Congruence

In this section, we introduce the notion of maximal L-fuzzy congruences and L-fuzzy maximal congruences on ADL A and discuss various properties of these. A proper congruence θ on an ADL A is said to be maximal if it is not properly contained in any proper congruence on A; (equivalently, for any congruence ψ on A, $\theta \subseteq \psi$ implies that either $\theta = \psi$ or $\psi = A \times A$). Also, an element $\alpha \neq 1$ in L is called a dual atom if there is no $\beta \in L$ such that $\alpha < \beta < 1$. Clearly, α is a dual atom if and only if α is a maximal element of

$L - \{1\}$. Here 1 stands for the largest element in L. Hereafter A stands for an ADL with a maximal element.

Definition 4.1. A proper L-fuzzy congruence of an ADL A is called a maximal L-fuzzy congruence on A if it is a dual atom in the lattice of the set of all proper L-fuzzy congruences on A under the point-wise partial ordering.

In the following, we determine all the maximal L-fuzzy congruences on A by establishing a one-to-one correspondence between maximal L-fuzzy congruences on A and pairs (θ, β), where θ is a maximal congruence on A and β a dual atom of L. First, we have the following corollary.

Corollary 4.2. *For any L-fuzzy congruence ϕ on ADL A and $\alpha \in L$, we define $\phi \vee \alpha$ by $(\phi \vee \alpha)(x, y) = \phi(x, y) \vee \alpha$, for all $(x, y) \in A \times A$. Then $\phi \vee \alpha$ is an L-fuzzy congruence on A.*

Theorem 4.3. *Let ϕ be a proper L-fuzzy congruence on A. Then ϕ is maximal if and only if $\phi = \beta_\theta$ for some maximal congruence θ on A and a dual atom β in L.*

Proof. Suppose that ϕ is a maximal L-fuzzy congruence on A. Let θ be the 1-cut ϕ_1 of ϕ; that is, $\theta = \{(a, b) \in A \times A : \phi(a, b) = 1\}$. Then θ is a congruence on A and, since ϕ is proper, $\theta \neq A \times A$. Clearly θ is not empty. We prove that ϕ assumes exactly one value other than 1. Since ϕ is proper, $\phi(a, b) < 1$ for some $a, b \in A$. Let $a, b, c, d \in A$ such that $\phi(a, b) < 1$ and $\phi(c, d) < 1$. Put $\gamma = \phi(a, b)$ and $\beta = \phi(c, d)$. Then $\phi \vee \gamma$ and $\phi \vee \beta$ are L-fuzzy congruences on A (by Theorem 4.2). Also, $(\phi \vee \gamma)(a, b) = \phi(a, b) \vee \gamma = \gamma \vee \gamma = \gamma < 1$ and $(\phi \vee \beta)(c, d) = \phi(c, d) \vee \beta = \beta < 1$ and hence $\phi \leq \phi \vee \beta < 1$. By the maximality of ϕ, we have $\phi = \phi \vee \gamma = \phi \vee \beta$. In particular, $\beta = \phi(c, d) = (\phi \vee \gamma)(c, d) = \phi(c, d) \vee \gamma = \beta \vee \gamma$ and $\gamma = \phi(a, b) = (\phi \vee \beta)(a, b) = \phi(a, b) \vee \beta = \gamma \vee \beta$ and hence $\gamma = \beta$. Therefore, ϕ assumes exactly one value, say β other than 1. Then

$$\phi(a, b) = \begin{cases} 1 & \text{if } (a, b) \in \theta (= \phi_1) \\ \beta & (a, b) \in \theta. \end{cases}$$

Therefore, $\phi = \beta_\theta$. If ψ is a proper congruence on A such that $\theta \subset \psi$, then $\phi = \beta_\theta \leq \beta_\psi \neq 1$ and by the maximality of ϕ, we get that $\beta_\theta = \beta_\psi$ and hence $\theta = \psi$. Thus, θ is a maximal congruence on A. Also, if $\gamma \in L$ with $\beta \leq \gamma < 1$, then $\phi = \beta_\theta \leq \gamma_\theta$, and again by the maximality of ϕ, $\beta_\theta = \gamma_\theta$ and hence $\gamma = \beta$. Thus, β is a dual atom in L. Conversely suppose that $\phi = \beta_\theta$, where θ is a maximal congruence on A and a dual atom

β in L. Since θ is a proper congruence, there exists $c, d \in A$ such that $(c, d) \notin \theta$ and hence $\phi(c, d) = \beta_\theta(c, d) = \beta < 1$. Therefore, ϕ is proper. Let η be an L-fuzzy congruence on A such that $\phi \leq \eta < 1$. Then $\theta = \phi_1 \subseteq \eta_1 \neq A \times A$. Since θ is maximal, we get that $\theta = \eta_1 = \{(a, b) \in A \times A : \eta(a, b) = 1\}$; that is, $\eta(a, b) = 1$, for all $(a, b) \in \theta$. If $(a, b) \notin \theta$, then $\beta = \beta_\theta(a, b) \leq \eta(a, b) < 1$ and hence $\beta = \eta(a, b)$, since β is a dual atom in L. Therefore, $\phi = \beta_\theta = \eta$. Thus ϕ is maximal. □

The following result is an immediate consequence of the above result.

Theorem 4.4. *$\phi(\theta, \beta)$ is a bijection correspondence between the set of pairs (θ, β), where θ is a maximal congruence on A and β a dual atom in L and the set of maximal L-fuzzy congruence on A.*

Proof. For any proper congruences θ and ψ on A and elements β and γ in L such that $\beta_\theta = \gamma_\psi$, $\quad (a, b) \in \theta \Rightarrow 1 = \beta_\theta(a, b) = \gamma_\psi(a, b)$
$\Rightarrow (a, b) \in \psi$
and $(a, b) \notin \theta \Rightarrow \beta = \beta_\theta(a, b) = \gamma_\psi(a, b)$
$\Rightarrow \gamma_\psi(a, b) = 1$
$\Rightarrow (a, b) \notin \psi$
and hence $\theta = \psi$ and $\gamma = \beta$. Therefore, the Theorem is an immediate consequence of 4.3. □

Definition 4.5. A proper L-fuzzy congruence ϕ on A is called an L-fuzzy maximal congruence on A if, for each $\beta \in L$, ϕ_β is either $A \times A$ or a maximal congruence on A.

The following theorem provides several examples of L-fuzzy maximal congruences on ADLs.

Theorem 4.6. *Let θ be a maximal congruences on A and $\beta < 1$ in L. Then β_θ is an L-fuzzy maximal congruence on A. In particular, χ_θ is an L-fuzzy maximal congruence on A.*

Proof. Put $\phi = \beta_\theta$. Then for any $x, y \in A$, we have

$$\phi(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \theta \\ \beta & \text{if } (x, y) \notin \theta. \end{cases}$$

Clearly, $\theta = \{(x, y) \in A \times A : \phi(x, y) = 1\}$. Since $\beta < 1$ and θ is proper, it follows that ϕ is proper. Now, for any $\gamma \in L$, we have

$\gamma \leq \beta \Rightarrow A \times A = \phi_\gamma \subseteq \phi_\beta \Rightarrow \phi_\beta = A \times A$

and $\gamma\beta \Rightarrow \theta \subseteq \phi_\gamma \neq A \times A$ (since $(x, y) \notin \theta \Rightarrow \phi(x, y) = \beta \Rightarrow (x, y) \notin \phi_\gamma$)

$\Rightarrow \phi_\gamma = \theta$ (since $\theta = \phi_1 \subseteq \phi_\gamma$).

Therefore, for each $\gamma \in L$, either $\phi_\gamma = A \times A$ or ϕ_γ is a maximal congruence on A. Hence the theorem. □

Theorem 4.7. *Let ϕ be an L-fuzzy maximal congruence on A. Then the following holds good.*

(1) *ϕ_1 is a maximal congruence on A*

(2) *There exists a largest element $\beta \in L$ such that $\phi_\beta = A \times A$*

(3) *ϕ assumes exactly two values.*

Proof. (1) Recall that $\phi_1 = \{(x, y) \in A \times A : \phi(x, y) = 1\}$. Since ϕ is proper, $\phi(x, y) \neq A \times A$, for some $x, y \in A$ and hence ϕ_1 is a proper congruence on A. Therefore, ϕ_1 is a maximal congruence on A.

(2) Since ϕ is proper, we have $\phi_1 \neq A \times A$ and $\phi_0 = A \times A$. Put $\beta = \vee\{\alpha \in L : \phi_\alpha = A \times A\}$. Then $\phi_\beta = \phi_{\vee\{\alpha \in L : \phi_\alpha = A \times A\}} = \bigcup_{\alpha \in L, \phi_\alpha = A \times A} = A \times A$. Clearly, for any $\alpha \in L$,

$\alpha \leq \beta \Rightarrow \phi_\beta \subseteq \phi_\alpha \Rightarrow A \times A \subseteq \phi_\alpha \Rightarrow \phi_\alpha = A \times A$

and $\phi_\alpha = A \times A \Rightarrow \alpha \leq \beta$.

(3) By(1), ϕ_1 is a maximal congruence on A. Put $\theta = \phi_1$. Then $\chi_\theta \leq \phi$ and hence $\phi = \beta_\theta$ for some $\beta \in L$. Since ϕ is proper and $\beta < 1$. Now,

$$\phi(a, b) = \begin{cases} 1 & \text{if } (a, b) \in \theta \\ \beta & \text{if } (a, b) \notin \theta. \end{cases}$$

Thus ϕ assumes exactly two values, namely 1 and β. □

Theorem 4.8. *A proper L-fuzzy congruence ϕ on A is an L-fuzzy maximal congruence on A if and only if $\phi = \beta_\theta$ for some $\beta < 1$ in L and maximal congruence θ on A.*

Proof. Suppose that ϕ is an L-fuzzy maximal congruence on A. Let $\theta = \phi_1$. Then by 4.7(1), θ is a maximal congruence on A. Also, by 4.7(3), ϕ assumes exactly two values. Clearly 1 is a value of ϕ. Let β be the other value of ϕ other than 1. Then $\beta < 1$ and for any $a, b \in A$

$$\phi(a, b) = \begin{cases} 1 & \text{if } (a, b) \in \theta \\ \beta & \text{if } (a, b) \notin \theta \end{cases}$$

and hence $\phi = \beta_\theta$. Converse follows from 4.6. □

Corollary 4.9. *Let θ be a proper congruence on A. Then θ is a maximal congruence on A if and only if χ_θ is an L-fuzzy maximal congruence on A.*

Corollary 4.10. *Every maximal L-fuzzy congruence is an L-fuzzy maximal congruence on A and L-fuzzy maximal congruence is an L-fuzzy prime congruence on A.*

The converse of the above corollary is not true; for, consider the following example.

Let $[0, 1]$, the closed unit interval in the real number system. Then L is a frame with respect to the usual ordering. For any $0 \leq \beta < 1$ in L and θ is a maximal congruence on an ADL A with a maximal element, β_θ is an L-fuzzy maximal congruence on A but not maximal, since L has no dual atom.

5 Conclusion

In this work, we study on the primeness and maximality of the set of all L-fuzzy congruence relations of a given ADL with truth values in a complete lattice L satisfying the infinite meet distributive law. In our future of work, we will focus on to investigate the L-fuzzy prime α-congruence and L-fuzzy prime α-ideals and study their relationship between these.

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